Rational points on twisted K3 surfaces and derived equivalences

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1 Previous work

Given a variety X (all varieties in this talk are assumed to be smooth projective) over a field k, we can construct the bounded derived category of coherent sheaves on X, $D^b(X)$. This is a k-linear triangulated category.

Motivating Question 1.1. *How much information about X does the derived category $D^b(X)$ contain?*

In fact, if ω_X is either ample or anti-ample, then $D^b(X)$ determines X up to isomorphism. (This is a theorem of Bondal and Orlov.) But this is not true in general: there exist non-isomorphic varieties X_1, X_2 such that $D^b(X_1) \cong D^b(X_2)$ as k-linear triangulated categories. Such equivalences are called *derived equivalences*, and they are always given by *Fourier-Mukai transforms*, namely the operation of pulling back to $D^b(X_1 \times X_2)$, tensoring with a suitable object in this category, and pushing forward.

Example 1.2. *Every abelian variety is derived equivalent to its dual, even though (in dimension greater than 1) they are generally not isomorphic. The equivalence $D^b(A) \cong D^b(A^\vee)$ is induced by the Poincaré bundle P on $A \times A^\vee$. Namely, $A^\vee$ is another way of writing $\mathrm{Pic}^0(A)$, the moduli space of degree-0 line bundles on A, and P is the universal line bundle over $A \times A^\vee$; its fiber over any $A \times \{\alpha\}$ is the line bundle represented by α.*

The case of K3 surfaces X_i provides a good testing ground for derived equivalences, on the one hand because the isomorphism $\omega_X \cong \mathcal{O}_X$ makes them slip through the gaps in the theorem of Bondal and Orlov, and on the other hand because K3 surfaces are nontrivial but tractable for lots of problems.

*Notes for a talk given in Berkeley's student arithmetic geometry seminar, organized by Martin Olsson. This talk is intended to summarize the paper of the same title by Ascher, Dasaratha, Perry, and Zhou. Other references: Hassett and Tschinkel, "Rational points on K3 surfaces and derived equivalence"; Hassett and Várilly-Alvarado, "Failure of the Hasse principle on general K3 surfaces"; Lieblich and Olsson, "Fourier-Mukai partners of K3 surfaces in positive characteristic".

Since $D^b(X)$ doesn't always determine X up to isomorphism (even for K3 surfaces), there are two paths to choose between: try to add some extra structure to $D^b(X)$ (such as filtrations and ample cones) so that it does determine X up to isomorphism, or study what properties of X are preserved by derived equivalences. The former path leads, for example, to the work of Lieblich and Olsson on derived Torelli theorems for K3 surfaces. Here we take the latter path. Specifically, we have:

Theorem 1.3. *(Lieblich-Olsson; Huybrechts) If k is a finite field, derived equivalences of K3 surfaces preserve point counts over all finite extensions of k, so in particular they preserve the existence of a k-point.*

Proof. (Rough idea.) Point counts over all finite extensions of k are encoded in the zeta function ζ_X. By the Lefschetz fixed-point formula (essentially the Weil conjectures), the ζ_X is determined by the traces of the Frobenius acting on $H^i_{\text{cris}}(X/W(k))$, where the only interesting piece is H^2. But this is determined by $D^b(X)$ via some motivic formalism. □

Theorem 1.4. *(Hassett, Tschinkel, and possibly Colliot-Thélène and Nikulin for $\mathbb{R}$) Derived equivalence of K3 surfaces preserves the existence of a rational point over $\mathbb{R}$. The same is true over p-adic fields assuming good reduction, or assuming ADE reduction if $p \geq 7$.*

Proof. (Rough idea.) For $\mathbb{R}$, there are some numerical invariants (related to the action of complex conjugation on $X_{\mathbb{C}}$) that characterize $X(\mathbb{R})^{\text{an}}$ up to diffeomorphism, and these invariants are preserved under derived equivalence. For p-adic fields, combine the result for finite fields with a Hensel's lemma argument. □

Hassett and Tschinkel then asked whether the same was true for twisted K3 surfaces. This paper, building on work of Hassett and Várilly-Alvarado, provides a negative answer; namely, it constructs two twisted K3 surfaces (X_1, α_1), (X_2, α_2) over $\mathbb{Q}$ with $D^b(X_1, \alpha_1) \equiv D^b(X_2, \alpha_2)$ but such that (X_2, α_2) has a $\mathbb{Q}$-point and (X_1, α_1) does not. Moreover, their example base-changes to examples over $\mathbb{Q}_2$ and over $\mathbb{R}$.

2 Brauer groups and Brauer-Manin obstructions

Let's first take a step back. Let k be a global field ($\mathbb{Q}$, if you prefer), and let X be a k-variety. If you want to check whether X has any k-points, the first thing to check is whether it has a k_v-point for all places v of k; equivalently, whether it has an $\mathbb{A}_k$-point. If so, the Hasse principle says it "should" have a k-point. But the Hasse principle can fail: nontrivial Brauer classes can obstruct the existence of k-points.

In the following definitions, we assume for convenience that our schemes are connected, so that locally free sheaves have constant rank.

Definition 2.1. *The* Brauer group *of a field k is the set of central simple algebras over k modulo the equivalence relation defined by $A \sim B$ if $A \otimes \mathrm{Mat}_m(k) \cong B \otimes \mathrm{Mat}_n(k)$ for some $m, n \geq 1$; this is a group under tensor product. (A* central simple algebra *over k is a finite-dimensional k-algebra with no nontrivial two-sided ideals and center equal to k.)*

Definition 2.2. *The* (Azumaya) Brauer group $\mathrm{Br}(X)$ *of a scheme X is defined analogously, with Azumaya algebras replacing central simple algebras and $\mathrm{Mat}_n(\mathcal{O}_X)$ replacing $\mathrm{Mat}_n(k)$. An* Azumaya algebra *is a locally free sheaf $\mathcal{A}$ on X that is étale-locally isomorphic to $\mathrm{Mat}_n(\mathcal{O}_X)$ for some $n > 0$.*

Definition 2.3. *The* (cohomological) Brauer group *of a scheme X (possibly $\operatorname{Spec} k$) is* $\mathrm{Br}'(X) = H^2_{ét}(X, \mathbb{G}_m)_{\mathrm{tors}}$.

Theorem 2.4. *For any scheme X, there is a natural injection from the Azumaya Brauer group $\mathrm{Br}(X)$ to $H^2_{ét}(X, \mathbb{G}_m)$. If X has finitely many connected components, $\mathrm{Br}(X)$ and therefore its image are torsion. This map is an isomorphism onto the torsion subgroup $\mathrm{Br}'(X)$ if X is quasiprojective over a noetherian ring. If X is nonsingular and quasiprojective over a field, then we have $\mathrm{Br}(X) \xrightarrow{\sim} H^2_{ét}(X, \mathbb{G}_m)_{\mathrm{tors}} = H^2_{ét}(X, \mathbb{G}_m)$.*

Today we will only care about smooth projective varieties over k, so we can safely use either definition.

Let's see how Brauer classes can obstruct the existence of k-points. Let $\alpha \in \mathrm{Br}(X)$. Then α can be "evaluated" at any k-point $x \in X(k)$ to produce a Brauer class $\alpha(x) \in \mathrm{Br}(k)$, and similarly $\alpha(x_v) \in \mathrm{Br}(k_v)$ for $x_v \in X(k_v)$. (Evaluation just means pulling back; i.e. pulling back Azumaya algebras or using the contravariance of $H^2_{ét}(-, \mathbb{G}_m)$.) Class field theory gives us an injection $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, which is an isomorphism for v non-archimedean. (At archimedean places, we have $\mathrm{Br}(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, and $\mathrm{Br}(\mathbb{C}) = 0$.) Then we have a short exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0, \tag{1}$$

where the map on the left is the obvious one given by tensoring central simple algebras from k to each k_v.

As a result, if $(x_v)_v$ is an adelic point of X, there is a nice way to check whether it has a chance to come from an actual k-point: map your element $\alpha \in \mathrm{Br}(X)$ to the middle term of the exact sequence above, and see whether it lies in the kernel of the map $\sum_v \mathrm{inv}_v$. If not, then there cannot be a k-point x that induces the $\mathbb{A}_k$-point $(x_v)_v$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(k) \arrow[r] & \bigoplus_v \mathrm{Br}(k_v) \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
& & \mathrm{Br}(X) \ni \alpha \arrow[u, "(x_v)_v"'] \arrow[ul, dashed, "\exists x?"] & &
\end{tikzcd}$$

Let $X(\mathbb{A}_k)^\alpha$ be the subset of $X(\mathbb{A}_k)$ consisting of α such that $\sum_v \alpha(x_v) = 0$, and more generally $X(\mathbb{A}_k)^S$ for $S \subset \mathrm{Br}(X)$. It may happen that $X(\mathbb{A}_k) \neq \emptyset$, but $X(\mathbb{A}_k)^S = \emptyset$ for some S. In this case, we say that S provides a *Brauer-Manin obstruction* to the existence of k-points of X.

3 Twisted K3 surfaces

Twisted K3 surfaces, twisted sheaves, and so on can be described cleanly in terms of $\mathbb{G}_m$-gerbes, but for our purposes it is fine to stay within the realm of schemes.

Definition 3.1. *A* twisted K3 surface *is a pair (X, α), where X is a K3 surface and $\alpha \in \mathrm{Br}(X)$ is a Brauer class. A k-*point *of (X, α) is a k-point x of X such that $\alpha(x) = 0$.*

Definition 3.2. *For $\alpha \in \mathrm{Br}(X)$, an α-*twisted sheaf *is a collection of sheaves on an étale cover $\{U_i\}$ of X equipped with isomorphisms on $U_i \times_X U_j$, such that the cocycle condition fails exactly by a cocycle represented by α.*

One can define quasicoherent and coherent twisted sheaves, and ultimately a derived category $D^b(X, \alpha)$ of coherent twisted sheaves. Like the ordinary (untwisted) derived category $D^b(X)$, this is a k-linear triangulated category, and there is a theory of twisted Fourier-Mukai equivalences, as Minseon discussed last month.

4 Construction

There are a few standard ways to construct K3 surfaces. For example, a smooth quartic surface in $\mathbb{P}^3$, a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$, and a smooth complete intersection of three quadrics in $\mathbb{P}^5$ are K3's. Another way to construct K3's is as a double cover of $\mathbb{P}^2$ ramified over a smooth sextic curve. Our examples will eventually be the latter kind, but we want to construct two of them in some coordinated way, so that their (twisted) derived categories have a chance to be equivalent.

Start with a double cover $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ ramified over a smooth curve of bidegree $(2, 2)$. Say the first copy of $\mathbb{P}^2$ has coordinates x_0, x_1, x_2 and the second y_0, y_1, y_2, so that the ramification divisor $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$ is cut out by a polynomial of degree 2 in the x_i's and degree 2 in the y_i's. We write this polynomial as $Ay_0^2 + By_0y_1 + Cy_0y_2 + Dy_1^2 + Ey_1y_2 + Fy_2^2$, where $A, \ldots, F$ are degree-2 polynomials in $k[x_0, x_1, x_2]$. Of course, we could switch the roles of the x_i and y_i and write the polynomial in terms of $A', \ldots, F' \in k[y_0, y_1, y_2]$ as well.

Let π_1 and π_2 be the projections $Y \to \mathbb{P}^2$. Each of these is a fibration by quadric surfaces. Next, for $i = 1, 2$, we take the relative Fano variety of lines F_i of π_i, whose fibers parametrize the lines on these quadric surfaces; namely, the fibers are $\mathbb{P}^1 \cup \mathbb{P}^1$ except along a divisor of $\mathbb{P}^2$ where the quadric surfaces are degenerate. Finally, we let X_i be the Stein factorization of $F_i \to \mathbb{P}^2$, which collapses each copy of $\mathbb{P}^1$ to a point. Thus X_i is a double cover of $\mathbb{P}^2$ ramified over a divisor, the discriminant divisor of $Y \to \mathbb{P}^2$, which turns out to have degree 6.

Explicitly, one can show that X_1 is cut out in the weighted projective space $\mathbb{P}(1, 1, 1, 3) = \operatorname{Proj} k[x_0, x_1, x_2, w]$ (with $\deg w = 3$) by the equation

$$w^2 = -\frac{1}{2} \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}. \tag{2}$$

(Of course, the analogous statement is true for X_2.) Then X_i is equipped with a Brauer class α_i induced by the $\mathbb{P}^1$-bundle $F_i \to X_i$. Explicitly, this can be realized as the class of the quaternion algebra over $K(X_i)$ with $i^2 = B^2 - 4AD$ and $j^2 = A$. (More precisely, this is an element of $\mathrm{Br}(K(X_i))$, which turns out to lie in the image of the natural injection $\mathrm{Br}(X_i) \to \mathrm{Br}(K(X_i))$ induced by the morphism of schemes $\operatorname{Spec} K(X_i) \to X_i$.)

With this setup, the authors show that $D^b(X_1, \alpha_1) \cong D^b(X_2, \alpha_2)$. Then they choose some explicit polynomials $A, \ldots, F$ making the following true.

First, X_1 has no $\mathbb{Q}$-points, even without twisting. This is proved by showing that for all $(x_v)_v \in X_1(\mathbb{A}_{\mathbb{Q}})$, we have $\alpha_1(x_v) = 0$ for all finite v, and $\alpha_1(x_\infty) = 1/2$. Since these don't add up to $0 \in \mathbb{Q}/\mathbb{Z}$, α_1 gives a Brauer-Manin obstruction to the existence of $\mathbb{Q}$-points on X_1. Calculating $\alpha_1(x_v)$ is immediate from a lemma for places of good reduction other than 2 and ∞, and it can be checked, with some calculation, using a proposition for bad places other than 2 and ∞. At ∞, it uses positive/negative definiteness conditions on the polynomials $A, \ldots, F$ as quadratic forms. At 2, it uses congruence conditions on the coefficients; formulating the "right" congruence conditions was the main contribution of Ascher, Dasaratha, Perry, and Zhou. (Precisely: the coefficients of $x_0^2y_0^2, x_1^2y_1^2, x_2^2y_2^2, x_0^2y_0y_1, x_1^2y_1y_2$, and $x_2^2y_2y_0$ must be odd, and the rest must be even.)

Second, X_2 has a $\mathbb{Q}$-point, namely $x = [1;1;1;0]$ in the coordinates on $\mathbb{P}(1,1,1,3)$. We want this to satisfy $\alpha_2(x) = 0 \in \mathrm{Br}(\mathbb{Q})$. Unfortunately this isn't actually true. But this problem can be solved easily: if we let $\beta = \alpha_2(x) \in \mathrm{Br}(\mathbb{Q})$, and embed this into each $\mathrm{Br}(X_i)$ by pulling back along the structure morphisms $X_i \to \operatorname{Spec}\mathbb{Q}$, then everything goes through for the adjusted Brauer classes $\alpha_i' = \alpha_i - \beta$. The counterexamples over $\mathbb{Q}_2$ and $\mathbb{R}$ are obtained by base-changing everything from $\mathbb{Q}$ to the respective completions, and modifying this last step of adjusting Brauer classes.